hep-th/0506126

Two-Loop Amplitudes of Gluons and Octa-Cuts in $\mathcal{N} = 4$ Super Yang-Mills

Evgeny I. Buchbinder[a] and Freddy Cachazo[b]

[a] *School of Natural Sciences, Institute for Advanced Study, Princeton NJ 08540 USA*

[b] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J 2W9, Canada*

After reduction techniques, two-loop amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory can be written in a basis of integrals containing scalar double-box integrals with rational coefficients, though the complete basis is unknown. Generically, at two loops, the leading singular behavior of a scalar double box integral with seven propagators is captured by a hepta-cut. However, it turns out that a certain class of such integrals has an additional propagator-like singularity. One can then formally cut the new propagator to obtain an octa-cut which localizes the cut integral just as a quadruple cut does at one-loop. This immediately gives the coefficient of the scalar double box integral as a product of six tree-level amplitudes. We compute, as examples, several coefficients of the five- and six-gluon non-MHV two-loop amplitudes. We also discuss possible generalizations to higher loops.

June 2005

1. Introduction

Recently there has been renewed interest in the perturbation expansion of $\mathcal{N} = 4$ super Yang-Mills. This was motivated by the discovery of a twistor string theory [1] that captures the perturbation theory of the maximally supersymmetric Yang-Mills theory (pMSYM). Twistor string theory has opened new avenues and has inspired new ideas for the computation of tree level amplitudes of gluons [2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19] and one-loop amplitudes of gluons in QCD [20,21,22], $\mathcal{N} = 1$ [23,24,25,26,27,28,29] and $\mathcal{N} = 4$ [30,31,32,33,34,35] super Yang-Mills. Before twistor string theory was introduced, the study of pMSYM at one-loop was mainly motivated by two facts: one is the decomposition of a QCD amplitude, A^{QCD}, with only a gluon running in the loop in terms of supersymmetric amplitudes and an amplitude with only a scalar running in the loop, $A^{\rm scalar}$, (see [36] for a review),

$$A^{\rm QCD} = A^{\mathcal{N}=4} - 4A^{\mathcal{N}=1}_{\rm chiral} + A^{\rm scalar} \tag{1.1}$$

where $A^{\mathcal{N}=4}$ has the full $\mathcal{N} = 4$ multiplet in the loop and $A^{\mathcal{N}=1}_{\rm chiral}$ only an $\mathcal{N} = 1$ chiral multiplet. The other motivation is a surprising proposal of Anastasiou, Bern, Dixon, and Kosower (ABDK) that two- (and, perhaps, higher-) loop amplitudes in pMSYM can be completely determined in terms of one-loop amplitudes [37]. This idea was inferred from studying collinear and IR singular behavior of the higher loop amplitudes. The conjecture is given in terms of normalized 2-loop amplitudes $M_n^{(2)} = A_n^{(2)}/A_n^{\rm tree}$ and in dimensional regularization, as follows

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f(\epsilon)M_n^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4 + \mathcal{O}(\epsilon). \tag{1.2}$$

This relation was explicitly verified for four-gluon amplitudes in [37] (see also section 7 of [38]). Also based on collinear limits [39], the schematic form of a relation analogous to (1.2) was proposed for higher loops [40]. Very recently, an explicit formula, analogous to (1.2), for the three-loop four-gluon amplitude was obtained and successfully verified in [41]. It is the aim of this paper to make some modest steps towards the calculation of higher loop amplitudes in pMSYM. The main motivation is to prepare the ground for future tests of the ABDK proposal. A proof of (1.2) would lead to the solution of pMSYM at two loops as a general solution to the one-loop problem can be obtained in terms of tree-level amplitudes by using quadruple cuts [34]. This is possible thanks to the cut constructibility of one-loop amplitudes in pMSYM proven in [42] and the decomposition

in terms of scalar box integrals, with rational functions as coefficients, also given in [42]. See also [30,31,32,33,35] for other techniques in pMSYM at one loop.

At two loops, a similar decomposition in terms of some given set of integrals is expected by using Passarino-Veltman or similar reduction procedures [43]. Unfortunately, the complete basis of two-loop integrals is currently unknown. However, scalar double box integrals are a natural ingredient of such a basis[1]. In this paper, we concentrate on the calculation of the coefficient of certain classes of planar scalar double box integrals. These are the integrals that arise in scalar field theory with a massless scalar running along internal lines and with the double-box structure depicted in fig. 1.

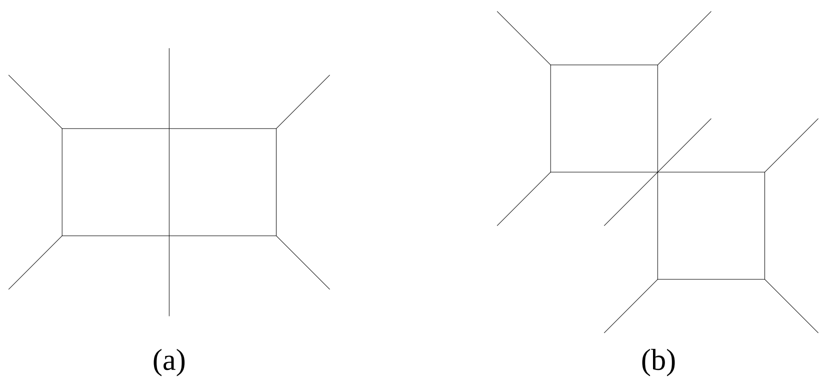

Fig. 1: The two possible different structures of planar scalar double box integrals. (a) Double boxes. (b) Split double boxes. Note that the momenta of the external lines is given by the sum of the momenta of external gluons.

The momenta of the external legs in fig. 1 are given by sums of momenta of external gluons.

We propose a method for computing the coefficient of any scalar double box integral given in fig. 1a when at least one of the two boxes has two adjacent massless three-particle vertices. We also give the form of the coefficient of any double box given in fig. 1b. In order to distinguish between the double boxes in fig. 1a and in fig. 1b we refer to the former simply as "double boxes" and the latter as "split double boxes".

Our original motivation was the successful use of quadruple cuts in the calculation of one-loop $\mathcal{N}=4$ amplitudes [34]. The basic idea is that at one-loop only scalar boxes contribute [42]. A quadruple cut singles out the contribution of a given scalar box and localizes the integration over the loop momentum. The combination of these two facts

[1] In fact, the four-gluon amplitude is given only in terms of scalar double boxes [44].

allows one to calculate any coefficient in terms of the product of four tree-level amplitudes [34]. Up to a numerical factor, every one-loop box coefficient is given by

$$B = \sum A^{\rm tree}_{(1)} A^{\rm tree}_{(2)} A^{\rm tree}_{(3)} A^{\rm tree}_{(4)}, \tag{1.3}$$

where the sum is over the solutions to the delta function equations and over all particles that can propagate in the loop. A straightforward application of this idea can be made for split double boxes (see fig. 1b). Again the idea is to cut all eight propagators, i.e., an octa-cut which localizes the two loop integrations and gives the coefficient as the product of seven tree-level amplitudes (up to a numerical factor)

$$B = \sum \prod_{i=1}^{7} A^{\rm tree}_{(i)}. \tag{1.4}$$

Naively, one might expect that the coefficient of double boxes in fig. 1a cannot be computed in a similar manner. The reason is that there are only seven propagators and a hepta-cut does not localize the integrals over the loop momenta.

A way to avoid the remaining integration arises in an unexpected manner. In studying singularities of Feynman integrals, one computes the discontinuity of an integral across a singularity by cutting propagators. When one cuts all propagators in a Feynman diagram one is computing the discontinuity across the leading singularity of the integral. However, at two (and higher) loops one finds a surprise when some of the external legs are massless. At two loops, if any of the two boxes in fig. 1a has at least two adjacent three-particle vertices (condition that is satisfied trivially for less than seven external gluons), then the integral has an extra propagator-like singularity beyond the naive leading singularity. The discontinuity across the new leading singularity is actually computed by an octa-cut[2]. This octa-cut precisely localizes the loop integrations and allows a straightforward computation of the coefficient as the product of six tree-level amplitudes. Up to a numerical factor, it is given by

$$B = \sum \prod_{i=1}^{6} A^{\rm tree}_{(i)}. \tag{1.5}$$

The only two-loop amplitude in pMSYM known in the literature is the four-gluon amplitude [44]. One reason is that very few double scalar box integrals are known explicitly

[2] For more general double boxes, there is also an extra singularity, these are known as second-type singularities [45]. They cannot easily be used to produce an octa-cut but they might give a generalization of it.

[46]. In particular, to our knowledge, not all double box integrals needed for a five-gluon amplitude are known. Nevertheless, we present the computation of several five-gluon and six-gluon non-MHV scalar double box integrals as illustrations of our technique.

This paper is organized as follows. In section 2, we review pMSYM at tree-, one-, two- and three-loop levels as well as the ABDK conjecture. In section 3, we show that the four-gluon amplitude of pMSYM can be found by using hepta-cuts. Even though the number of cut propagators is less than the number of integration variables, the integrand turns out not to depend on the loop momenta and can be pulled out of the integral. In section 4, we demonstrate that a certain class of double-box configurations admit an extra propagator-type singularity. Cutting this singularity allows us to write a universal formula for many double-box coefficients. In section 5, we illustrate our technique via various examples including non-MHV amplitudes. In section 6, we discuss applications to three- and higher-loop amplitudes. In particular, we show that by studying singularities, it is possible to realize that the basis of integrals has to contain integrals with some non-trivial factors in the numerator. This is in agreement with results of [44].

Throughout the paper, we use the following notation and conventions along with those of [1] and the spinor helicity-formalism [47,48,49]. A external gluon labeled by i carries momentum K_i. Since $K_i^2=0$, it can be written as a bispinor $(K_i)_{a\dot{a}}=\lambda_{i\ a}\widetilde{\lambda}_{i\ \dot{a}}$. Inner product of null vectors $p_{a\dot{a}}=\lambda_a\widetilde{\lambda}_{\dot{a}}$ and $q_{a\dot{a}}=\lambda'_a\widetilde{\lambda}'_{\dot{a}}$ can be written as $2p\cdot q=\langle\lambda,\lambda'\rangle[\widetilde{\lambda},\widetilde{\lambda}']$, where $\langle\lambda,\lambda'\rangle=\epsilon_{ab}\lambda^a\lambda'^b$ and $[\widetilde{\lambda},\widetilde{\lambda}']=\epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}^{\dot{a}}\widetilde{\lambda}'^{\dot{b}}$. Other useful definitions are:

$$
\begin{aligned}
K_{i,\ldots,j} &\equiv K_i+K_{i+1}+\cdots+K_j\\
K_i^{[r]} &\equiv K_i+K_{i+1}+\cdots+K_{i+r-1}\\
\langle i|\sum_r K_r|j] &\equiv \sum_r \langle i\ r\rangle[r\ j]\\
\langle i|(\sum_r K_r)(\sum_s K_s)|j\rangle &\equiv \sum_r\sum_s \langle i\ r\rangle[r\ s]\langle s\ j\rangle\\
[i|(\sum_r K_r)(\sum_s K_s)|j] &\equiv \sum_r\sum_s [i\ r]\langle r\ s\rangle[s\ j]\\
\langle i|(\sum_r K_r)(\sum_s K_s)(\sum_t K_t)|j] &\equiv \sum_r\sum_s\sum_t \langle i\ r\rangle[r\ s]\langle s\ t\rangle[t\ j]
\end{aligned}
\tag{1.6}
$$

where addition of indices is always done modulo n.

2. Review of $\mathcal{N}=4$ Amplitudes

In this paper we consider amplitudes of gluons in $\mathcal{N}=4$ super-Yang-Mills. Each gluon carries the following information: momentum $p_{a\dot{a}}$, polarization vector $\epsilon_{a\dot{a}}$ and color index a. The color structure can be striped out by a color decomposition [50,51,52,53]. Here we only consider the leading color or planar part of the amplitudes. The information in momentum and polarization vectors can be encoded in terms of spinors λ, $\widetilde{\lambda}$ and the helicity of the gluon h.

2.1. Tree-Level $\mathcal{N}=4$ Amplitudes

At tree-level, the leading color approximation is exact. An amplitude is given by

$$A_{(\{p_i,\epsilon_i,a_i\})} = g_{\rm YM}^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}\,(T^{a_{\sigma(1)}}\ldots T^{a_{\sigma(n)}}) A_{(\{\lambda_{\sigma(1)},\widetilde{\lambda}_{\sigma(1)},h_{\sigma(1)}\},\ldots,\{\lambda_{\sigma(n)},\widetilde{\lambda}_{\sigma(n)},h_{\sigma(n)}\})}. \tag{2.1}$$

Here we are suppressing a delta function that imposes momentum conservation.

It is convenient to denote the set of data $\{\lambda_i,\widetilde{\lambda}_i,h_i\}$ by i^{h_i}, where $h_i=\pm$ is the helicity of the i^{th} gluon. The amplitudes on the right hand side of (2.1) are known as leading color partial amplitudes and are computed from color-ordered Feynman rules. One can study a given order $A(1^{h_1},\ldots,n^{h_n})$ and the rest can be obtained by application of permutations, σ.

The partial amplitude $A(1^{h_1},\ldots,n^{h_n})$ can be computed using a variety of methods (see [36] for a nice review on many of the techniques developed in the 80's and 90's). More recently, two new techniques became available, namely, MHV diagrams [2] and the BCFW recursion relations [14,15]. The latter is a set of quadratic recursion relations for on-shell physical partial amplitudes of gluons. For a recent review see [54].

2.2. One-Loop $\mathcal{N}=4$ Amplitudes

Amplitudes of gluons at one-loop admit a color decomposition [50,51,52,53,55] with single and double trace contributions. As mentioned in the introduction we will only concentrate on the leading color partial amplitudes[3].

[3] It is interesting to note that since for $\mathcal{N}=4$ SYM all particles in the loop are in the adjoint representation, all sub-leading color amplitudes are given as linear combinations of the planar ones with permutations of the gluon labels (See section 7 of [42] for a proof.)

One-loop amplitudes of gluons in supersymmetric theories are four-dimensional cut-constructible [42,56]. This means that they can be completely determined by their finite branch cuts and discontinuities. $\mathcal{N} = 4$ amplitudes are even more special. Reduction techniques [43] can be used to express these amplitudes in terms of scalar box integrals [42]. These are one-loop box Feynman integrals in a scalar field theory where a massless scalar runs in the loop,

$$\mathcal{I} = \int d^4\ell \frac{1}{(\ell^2 + i\epsilon)((\ell - k_1)^2 + i\epsilon)((\ell - k_1 - k_2)^2 + i\epsilon)((\ell + k_4)^2 + i\epsilon)} \tag{2.2}$$

where k_1, k_2, k_3, k_4 are the external momenta at each vertex. They are not independent since by momentum conservation $k_3 = -(k_4 + k_1 + k_2)$. Note that the integral (2.2) is singular when at least one k_i is a null vector. Therefore, we should specify a regularization procedure, like dimensional regularization. However, we will be considering cuts that are finite and do not depend on the regularization procedure. Since $A(1, \ldots, n)$ is color-ordered, each k can only be the sum of consecutive momenta of external gluons. Moreover, since we only consider the planar contributions, we can define a given contribution by specifying i, j, k, l such that $k_1 = K_i + \ldots + K_{j-1}$, $k_2 = K_j + \ldots + K_{k-1}$ and $k_3 = K_k + \ldots + K_{l-1}$. The reduction procedure then gives for the amplitude an expansion of the form [42]

$$A(1, \ldots, n) = \sum_{1<i<j<k<m<n} B_{ijkl}\ \mathcal{I}_{(K_i + \ldots + K_{j-1}, K_j + \ldots + K_{k-1}, K_k + \ldots + K_{l-1})}, \tag{2.3}$$

where the coefficients B_{ijkm} are rational functions of the spinor products. Since all scalar box integrals are known explicitly, the problem of computing $A(1, \ldots, n)$ is reduced to that of computing the coefficients B_{ijkl}.

A general formula for the coefficients B_{ijkl} was found in [34] in terms of products of tree level amplitudes. Let us review the derivation of the formula because the idea is useful in the analysis at higher loops. If we think of the scalar box integrals as an independent basis[4] of some vector space we can interpret $A(1, \ldots, n)$ as a general vector. All we need to do is to find a way to project $A(1, \ldots, n)$ along the space spanned by a given scalar box integral $\mathcal{I}$. From the definition of $\mathcal{I}$ in (2.2) we see that each integral is uniquely determined once its four propagators are given. It is natural to think that the way to determine the coefficient B is by looking at the region of integration where all four propagators become

[4] The notion of independence is the equivalent of cut constructibility of the amplitude.

singular. In fact, the integral obtained by cutting, i.e., by dropping the principal part of all four propagators computes the discontinuity of the given scalar box integral across a singularity which is unique to it.

The set of four equations that gives ℓ is the following

$$\ell^2 = 0, \qquad (\ell - k_1)^2 = 0, \qquad (\ell - k_1 - k_2)^2 = 0, \qquad (\ell + k_4)^2 = 0. \tag{2.4}$$

A little exercise shows that these equations do not have a solution if ℓ is a real vector in Minkowski space for general external momenta. The way out of this problem is to complexify all momenta and make a Wick rotation to $(- - ++)$ signature. In the new signature the delta functions are still well defined and there are always solutions to (2.4).

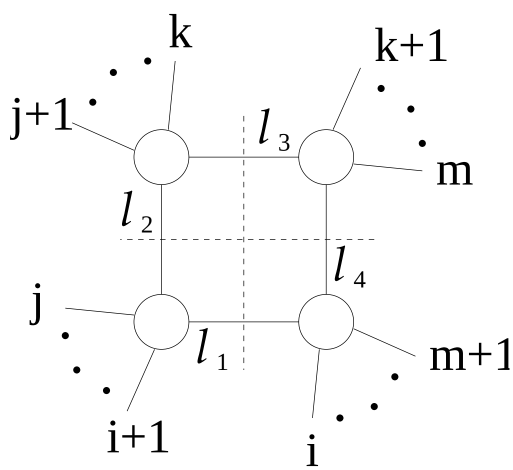

Fig. 2: A quadruple cut diagram. Momenta in the cut propagators flows clockwise and external momenta are taken outgoing. The tree-level amplitude $A_1^{\rm tree}$, for example, has external momenta $\{i+1, ..., j, \ell_2, \ell_1\}$.

One can also look at the same regime on the left hand side of (2.3) by considering only Feynman diagrams that posses the four propagators entering in (2.4). Summing over them one finds the following equation

$$\int d\mu \sum_J n_J A^{tree}_{(1)} A^{tree}_{(2)} A^{tree}_{(3)} A^{tree}_{(4)} = B_{ijkm} \int d\mu \tag{2.5}$$

where sum over J represents a sum over all possible particles in the $\mathcal{N} = 4$ multiplet. The measure $d\mu$ is the same one both sides of the integrals,

$$d\mu = d^4\ell \; \delta^{(+)}(\ell^2) \; \delta^{(+)}((\ell - k_1)^2) \; \delta^{(+)}((\ell - k_1 - k_2)^2) \; \delta^{(+)}((\ell + k_4)^2), \tag{2.6}$$

and the tree-level amplitudes are defined as follows (see fig. 2)

$$\begin{aligned} A^{tree}_{(1)} =& A(-\ell_1, i+1, i+2, \ldots, j-1, j, \ell_2), \quad A^{tree}_{(2)} = A(-\ell_2, j+1, j+2, \ldots, k-1, k, \ell_3), \\ A^{tree}_{(3)} =& A(-\ell_3, k+1, k+2, \ldots, m-1, m, \ell_4), \; A^{tree}_{(4)} = A(-\ell_4, m+1, m+2, \ldots, i-1, i, \ell_1). \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} &\ell_1 = \ell, \; \ell_2 = \ell - k_1, \; \ell_3 = \ell - k_1 - k_2, \; \ell_4 = \ell + k_4, \; k_1 = K_{i+1} + \ldots + K_j, \\ &k_2 = K_{j+1} + \ldots + K_k, \; k_3 = K_{k+1} + \ldots + K_m, \; k_4 = K_{m+1} + \ldots + K_i. \end{aligned} \tag{2.8}$$

The integral $\int d\mu$ is just given by a Jacobian $1/\sqrt{\Delta}$. This Jacobian cancels on both sides since the integral is localized by the delta functions and the coefficient is given by [34]

$$B_{ijkl} = \frac{1}{|\mathcal{S}|} \sum_{\mathcal{S},J} n_J A^{tree}_{(1)} A^{tree}_{(2)} A^{tree}_{(3)} A^{tree}_{(4)}. \tag{2.9}$$

Here $\mathcal{S}$ is the set of solutions to the conditions imposed by the delta functions, and $|\mathcal{S}|$ is the number of solutions. The sum also involves a sum over all possible particles that can propagate in the loop. For further details and many examples we refer to [34]. Even though the Jacobian did not play an important role for the quadruple cut technique at one-loop, it is crucial for the two-loop analysis we carry out in section 4.1. For this reason let us write it down for future reference

$$\Delta = s^2 t^2 - 2st(k_1^2 k_3^2 + k_2^2 k_4^2) + (k_1^2 k_3^2 - k_2^2 k_4^2)^2 \tag{2.10}$$

with $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$.

2.3. Two-Loop $\mathcal{N} = 4$ Amplitudes

At two loops, only the four-gluon amplitude has been computed [44]. The $\mathcal{N} = 4$ calculation was the first full two-loop amplitude of gluons ever computed. The answer is given as a linear combination of double box scalar integrals with coefficients that are rational function of the spinor variables. A double box scalar integral is the analog of the one-loop scalar box integral introduced above, more explicitly,

$$\begin{aligned} \mathcal{I}(k_1, \ldots, k_6) = &\int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + i\epsilon)((p-k_1)^2 + i\epsilon)((p-k_1-k_2)^2 + i\epsilon)} \times \\ &\int \frac{d^4 q}{(2\pi)^4} \frac{1}{((p+q+k_6)^2 + i\epsilon)(q^2 + i\epsilon)((q-k_5)^2 + i\epsilon)((q-k_4-k_5)^2 + i\epsilon)}. \end{aligned} \tag{2.11}$$

This integral is UV finite but it might have IR divergences when some k's are null vectors. Again, as in the one-loop case, one has to choose a regularization procedure but we do not do so because we only discuss finite cuts. The planar contribution to the four-gluon amplitude is [44]

$$A_4^{2-loop}(K_1,K_2,K_3,K_4) = A_4^{tree}\ s\ t\,(s\,\mathcal{I}(K_1,K_2,0,K_3,K_4,0) + t\,\mathcal{I}(K_4,K_1,0,K_2,K_3,0)) \tag{2.12}$$

where $s=(K_1+K_2)^2$ and $t=(K_2+K_3)^2$. This was computed by using the unitarity-based method [42,56,57,58,59]. It is very important to mention that the double box scalar integral (2.11) is not known in general but explicitly formulas exists in dimensional regularization when $k_3 = k_6 = 0$ and at least three of the other k_i's are null vectors [46].

2.4. ABDK Conjecture

As mentioned in the introduction, one of the motivations of this work is to prepare the ground for a more extensive test of the ABDK conjecture. This conjecture asserts that the planar limit of L-loop amplitudes in $\mathcal{N}=4$ SYM is determined iteratively, i.e., as a function of l-loop amplitudes with $l<L$.

Let us make this more precise. Here we follow [37] and [40] where the original proposal was made. Consider the function

$$M_n^{(L)}(1,2,\ldots,n) = \frac{A^{L-loop}(1,2,\ldots,n)}{A^{tree}(1,2,\ldots,n)} \tag{2.13}$$

then the ABDK conjecture states that

$$M_n^{(L)} = P_L(M_n^{(1)},\ldots,M_n^{(L-1)}) \tag{2.14}$$

where $P_L(x_1,\ldots,x_{L-1})$ is a certain polynomial of degree L and independent of the helicity configuration. The explicit form of (2.14) at two loops was given in [37] in terms of the function $f(\epsilon) = (\psi(1-\epsilon)-\psi(1))/\epsilon$, where the digamma function is defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$, as follows

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f(\epsilon)M_n^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4. \tag{2.15}$$

This conjecture was explicitly checked for four-gluon amplitudes. Very recently, the form of the polynomial in (2.14) was obtained for the three-loop four-gluon amplitude in [41].

One of the impressive predictions of the conjecture is a relation between the finite remainders which are defined at $\epsilon = 0$. At two loops, one introduces the universal singular function $C_n(\epsilon)^{(2)}$ [60,37] which contains the infrared singularities and does not depend on the helicity configuration since it is normalized by the tree-level amplitude. Defining the finite remainder as

$$F_n^{(2)}(\epsilon) = M_n^{(2)}(\epsilon) - C_n^{(2)}(\epsilon), \tag{2.16}$$

one can write a finite (as $\epsilon \to 0$) analog of (2.15) as follows [37]

$$F_n^{(2)}(0) = \frac{1}{2}\left(F_n^{(1)}(0)\right)^2 - \zeta_2 F_n^{(1)}(0) - \frac{1}{4}\left(\frac{11n}{8} + 5\right)\zeta_4. \tag{2.17}$$

Recall that at one-loop $F_n^{(1)}(0)$ can have at most dilogarithms, while $F_n^{(2)}(0)$ can have higher polylogarithms. This means that very non-trivial cancelations must happen. These cancelations were found to occur for $n = 4$ between terms coming from the two integrals in (2.12) and involved many polylogarithmic identities [37]. In the recent paper [41], an impressive formula for the all loop finite remainder of MHV amplitudes was also presented. The formula is given in a kind of generating function structure

$$1 + \sum_{L=1}^{\infty} a^L F_n^{(L)}(0) = \exp\left[\frac{1}{4}\gamma_K F_n^{(1)}(0) + C\right] \tag{2.18}$$

where a is basically the 't Hooft coupling, γ_K is the universal soft anomalous dimension and C is a function that admits a power series representation in a. γ_K and C are known up to the order needed to obtain the three-loop term[5].

3. Four-Gluon Two-Loop Amplitudes and Hepta-Cuts

In this section, we consider hepta-cuts of the two-loop four-gluon leading partial amplitude. This section can be viewed as a warm-up section where we introduce relevant notation and do some sample calculations which will be used in the rest of the paper. It is enough to consider $A_4^{2-loop}(1^-, 2^-, 3^+, 4^+)$ as all other A_4^{2-loop} with different helicity

[5] It is important to mention that there is no canonical definition of the finite remainder F_n. In fact, the definition of finite remainder used in (2.17) differs from that used in (2.18). For more details see [41]. We thank Z. Bern and L. Dixon for useful discussions on this point.

assignments can be obtained from this one by Ward identities. The leading partial amplitude was first computed in [44]. As reviewed in section 2.3, the amplitude can be presented as a linear combination of two scalar double-box integrals (see fig. 1a)

$$\mathcal{I} = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{1}{p^2(p-K_1)^2(p-K_1-K_2)^2(p+q)^2q^2(q-K_4)^2(q-K_3-K_4)^2}, \tag{3.1}$$

where K_i are the four external gluon momenta, with rational coefficients. All external momenta are assumed to be outgoing. The integral (3.1) has seven propagators, hence it is natural to consider hepta-cuts. It turns out that the coefficients can easily be found from hepta-cuts when the loop momenta are analytically continued to signature $(--++)$ or complexified. In the present case, there are two independent coefficients as well as two independent hepta-cuts. We refer to them as the s-channel cut and the t-channel cut. The corresponding coefficients will be denoted as c_s and c_t. Let us start with the cut in the s-channel. In this case, there are six different helicity configurations. For all of them, only gluons can propagate in both loops. A sample helicity configuration is shown in fig. 3. In this paper, for simplicity, we depict tree level amplitudes as points. Since all propagator are cut, there is no need to indicate a cut by a dash line and we choose to omit them [6].

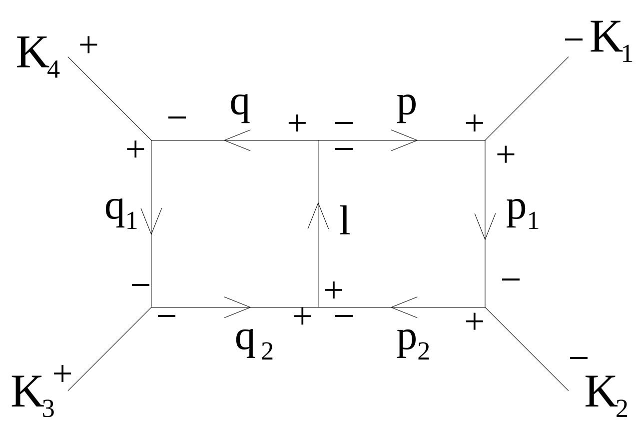

Fig. 3: A sample hepta-cut in the s-channel. Tree level amplitudes are depicted as points. All propagators are cut and therefore we omit the dashed lines used in fig. 2.

[6] Note that conventions in fig. 3 are different from those used in fig. 2 where all tree level amplitudes are denoted by blobs and cuts are indicated by dashed lines.

The rational coefficient c_s is then given by

$$c_s = \frac{i^7 \sum_{I=1}^{6} \int d\mu (A_{(1)}^{tree} A_{(2)}^{tree} A_{(3)}^{tree} A_{(4)}^{tree} A_{(5)}^{tree} A_{(6)}^{tree})_I}{\int d\mu}, \tag{3.2}$$

where by $A_{(i)}^{tree}$ we denote tree amplitudes at each of the six vertices, the integration measure $d\mu$ is given by

$$\begin{aligned} d\mu =& \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \delta(p^2) \delta((p-K_1)^2) \delta((p-K_1-K_2)^2) \delta((p+q)^2) \\ & \delta(q^2) \delta((q-K_4)^2) \delta((q-K_3-K_4)^2) \end{aligned} \tag{3.3}$$

and in the numerator we sum over the product of the six tree level amplitudes corresponding to a given helicity configuration. The factor i^7 comes from the seven propagators. It seems that since the number of delta functions is less than the number of integration variables, the integral does not localize and one integration has to be performed. However, it turns out that the integrand can be simplified in such a way that the dependence on the loop momenta drops out and we are left with the integral of the measure which cancels out according to eq. (3.3).

In the discussion of one-loop amplitudes in section 2.2, we mentioned that the momentum ℓ has to be complexified in order to find solutions to the four equations from the cut propagators. Making ℓ complex also has as a byproduct the fact that three-particle vertices on-shell do not have to vanish. Tree-level three-gluon amplitudes with helicities $(--+)$ or $(++-)$ are given respectively by [61,62]

$$A_3^{tree}(p^-, q^-, r^+) = i \frac{\langle p\ q\rangle^3}{\langle q\ r\rangle \langle r\ p\rangle}, \quad A_3^{tree}(p^+, q^+, r^-) = -i \frac{[p\ q]^3}{[q\ r][r\ p]}. \tag{3.4}$$

In Minkowski space, λ_p and $\widetilde{\lambda}_p$ are related to each other as $\widetilde{\lambda}_p = \pm \overline{\lambda}_p$. This means that if $p \cdot q = 0$, which follows from momentum conservation at the vertex, then both $\langle \lambda_p\ \lambda_q \rangle = 0$ and $[\widetilde{\lambda}_p\ \widetilde{\lambda}_q] = 0$. This implies that both amplitudes in (3.4) vanish. If we complexify the momenta, then the equation $p \cdot q = 0$ has two independent solutions. We have that either $\langle \lambda_p\ \lambda_q \rangle = 0$ or $[\widetilde{\lambda}_p\ \widetilde{\lambda}_q] = 0$. That is either λ_p and λ_q are proportional or $\widetilde{\lambda}_p$ and $\widetilde{\lambda}_q$ are proportional. Also note that momentum conservation implies that $p \cdot q = p \cdot r = q \cdot r = 0$. This means that either three λ's are proportional or all three $\widetilde{\lambda}$'s are proportional. Therefore, for every $(++-)$ tree level amplitude we choose all λ's to be proportional. Similarly, for every $(--+)$ tree level amplitude we choose all three $\widetilde{\lambda}$'s to be proportional.

Explicit calculations, considered for one of the helicity configurations in some detail below, show that every helicity configuration gives the same contribution equal to

$$-A_4^{tree} s^2 t \int d\mu, \tag{3.5}$$

where A_4^{tree} is the tree-level four-gluon amplitude

$$A_4^{tree}(1^-, 2^-, 3^+, 4^+) = i \frac{\langle 1\ 2\rangle^3}{\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 1\rangle}, \tag{3.6}$$

and

$$s = (K_1 + K_2)^2,\ t = (K_2 + K_3)^2. \tag{3.7}$$

Note that the integral in (3.5) cancels against the denominator in (3.2). The coefficient 6 in the numerator will also cancel. The reason is the following. In the denominator in (3.2), we have to sum over all different solutions to the delta-function conditions. It is easy to realize that in this particular case the number of different solutions equals the number of helicity configurations. Thus, each term in the numerator in (3.2) picks one of the six solutions whereas in the denominator we sum over all the six solutions. As a result, we obtain

$$c_s = -A_4^{tree} s^2 t. \tag{3.8}$$

This coincides with the corresponding coefficient found in [44].

Let us consider the helicity configuration shown in fig. 3 in some detail. The analysis of the remaining five configurations is completely analogous. Consider the integrand as the product of six tree-level amplitudes

$$\begin{gathered} -i(A_{(1)}^{tree} A_{(2)}^{tree} A_{(3)}^{tree} A_{(4)}^{tree} A_{(5)}^{tree} A_{(6)}^{tree})_1 = -i \left(\frac{[p_1, p]^3}{[p, 1][1, p_1]} \right) \left(\frac{\langle p_1, 2\rangle^3}{\langle 2, p_2\rangle\langle p_2, p_1\rangle} \right) \\ \left(\frac{[q_2, l]^3}{[l, p_2][p_2, q_2]} \right) \left(\frac{\langle p, l\rangle^3}{\langle l, q\rangle\langle q, p\rangle} \right) \left(\frac{[q_1, 4]^3}{[4, q][q, q_1]} \right) \left(\frac{\langle q_1, q_2\rangle^3}{\langle q_2, 3\rangle\langle 3, q1\rangle} \right). \end{gathered} \tag{3.9}$$

Next, simplify this expression by using momentum conservation. For example, the product of $[p_1, p]$ and $\langle p_1, 2\rangle$ can be simplified as follows

$$[p_1, p]\langle p_1, 2\rangle = -\langle 2|p_1|p] = \langle 2|K_1|p] = -\langle 1\ 2\rangle[1\ p]. \tag{3.10}$$

Then the product of the first four factors in (3.9) becomes

$$\langle 1\ 2\rangle^2 [q, q_2]^2. \tag{3.11}$$

After using momentum conservation along the lines of eq. (3.10), one finds

$$-i(A^{tree}_{(1)}A^{tree}_{(2)}A^{tree}_{(3)}A^{tree}_{(4)}A^{tree}_{(5)}A^{tree}_{(6)})_1 = is^2\frac{\langle 1\ 2\rangle^2[3\ 4]}{\langle 3\ 4\rangle} = -A^{tree}_4 s^2 t. \tag{3.12}$$

Note that this expression does not depend on the loop momenta and, thus, can be pulled out of the integration.

Now we consider the hepta-cut in the t-channel. Here we have ten helicity configurations. Note that in this case the number of helicity configurations does not equal the number of solutions of the delta-function equations. By a solution we mean a choice whether all λ's are proportional or $\widetilde{\lambda}$'s are proportional at each of six three gluon vertices. However, among the ten configurations, there are different configurations for which the choices of whether λ's or $\widetilde{\lambda}$'s are proportional are exactly the same. A solution then means summing up over such configurations. In this case, there are two helicity configurations corresponding to actual solutions and the remaining eight ones break up in pairs. The sum of the two helicity configurations in each pair corresponds to a solution of the delta-function conditions. Overall, we have six improved helicity configurations, each corresponding to an independent solution to the delta-function conditions. All paired up configurations involve fermions and scalars running in one of the loops and summation over the two configurations in a given pair provides a significant simplification. The coefficient c_t is given by

$$c_t = -\frac{i\sum_{I=1}^{10}\int d\mu(A^{tree}_{(1)}A^{tree}_{(2)}A^{tree}_{(3)}A^{tree}_{(4)}A^{tree}_{(5)}A^{tree}_{(6)})_I}{\int d\mu}, \tag{3.13}$$

where the sum is over all ten configurations, or over the six improved configurations, each corresponding to an actual solution of the delta-function equations. As before, all six improved configurations give the same contribution

$$-A^{tree}_4 st^2. \tag{3.14}$$

Since each improved configuration corresponds to a solution to the delta-function equations, the factor 6 cancels out. As a result we obtain

$$c_t = -A^{tree}_4 st^2. \tag{3.15}$$

This coincides with the corresponding coefficient from [44]. As an example, let us consider the two helicity configurations shown in fig. 4.

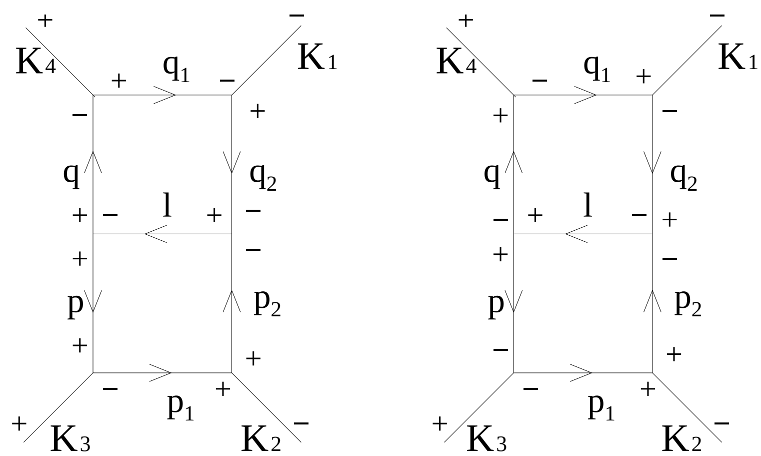

Fig. 4: Examples of hepta-cuts in the t-channel that correspond to the same solution of the delta function constraints.

Note that for both double boxes the choices whether all λ's or all $\widetilde{\lambda}$'s are proportional are exactly the same at every vertex. Therefore, it is the sum of these two diagrams that corresponds to one of the six improved helicity configurations. Both double boxes in fig. 4 involve gluons, fermions and scalars running in one of the loops and only gluons running in the remaining loop. The necessary tree level amplitudes are given by

$$A_3^{tree}(p^-,q^-,r^+) = i\frac{\langle p\ q\rangle^3}{\langle q\ r\rangle\langle r\ p\rangle}\left(\frac{\langle q\ r\rangle}{\langle q\ p\rangle}\right)^a, \quad A_3^{tree}(p^+,q^+,r^-) = -i\frac{[p\ q]^3}{[q\ r][r\ p]}\left(\frac{[q\ r]}{[q\ p]}\right)^a, \tag{3.16}$$

where $a = 0$ for gluons, $a = 1$ for fermions and $a = 2$ for scalars. After summation over the two configurations, we obtain

$$-i\frac{(\alpha-\beta)^4}{\gamma}, \tag{3.17}$$

where

$$\begin{aligned} &\alpha = \langle q_1\ 1\rangle\langle q_2\ p_2\rangle[p\ q][4\ q_1], \quad \beta = \langle 1\ q_2\rangle\langle p_2\ l\rangle[l\ p][q\ 4] \\ &\gamma = \langle q_2\ q_1\rangle\langle l\ q_2\rangle[q\ l][q_1\ q]\langle 1\ q_2\rangle\langle p_2\ l\rangle[l\ p][q\ 4]\langle q\ 1\rangle\langle q_2 p_2\rangle[4\ q_1][p\ q]. \end{aligned} \tag{3.18}$$

By using momentum conservation along the lines of eq. (3.10), we can simplify (3.18) to obtain $-A_{tree}st^2$. The integral of the measure factors out and cancels against the denominator in (3.13) to give (3.15).

Thus, we find that the coefficients of the four-gluon amplitude double boxes can be calculated by studying hepta-cuts. Of course, this is not enough to claim that this is the full answer. One still has either to prove that the answer has all the correct discontinuities across all branch cuts, which was done in [44], or to prove that the basis of integrals is

given entirely by double boxes. Unfortunately, the basis of integrals is not known at two loops.

For four gluons, even though the number of integration variables is greater that the number of the delta-functions in a hepta-cut, no integration has to be performed. We find that this is not the case if the number of external gluons is greater than four. We will see in the next section that already in the case of five-gluon amplitude, the product of the corresponding tree-level amplitudes does depend on the loop momenta and cannot be pulled out of the integral.

4. Octa-Cuts of Two-Loop Amplitudes

In the introduction we distinguished between two different kinds of double box scalar integrals. In the first class, the two boxes share a propagator while in the second class they only share a vertex, see fig. 1a and fig. 1b respectively. In this section, we show how one can use octa-cuts to compute the coefficient of a certain subset of the first class and the coefficients of all integrals of the second class, which we called split double boxes.

4.1. Double-Box Scalar Integrals

Let us start with the double boxes that have seven propagators. We will show that when at least one of the two boxes has two adjacent three particle vertices then there is an extra propagator-like singularity that can be cut. This produces one more delta-function which together with the hepta-cut of the previous section completely localizes the cut integral. Even though we concentrate only on the planar configurations, exactly the same logic can be applied for non-planar configurations as well. Consider an arbitrary double-box configuration shown in fig. 5. The corresponding hepta-cut integral is

$$\mathcal{I} = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\delta(p^2)\delta((p-k_1)^2)\delta((p-k_1-k_2)^2)\delta((p+q+k_6)^2)\delta(q^2)\delta((q-k_5)^2)\delta((q-k_4-k_5)^2). \tag{4.1}$$

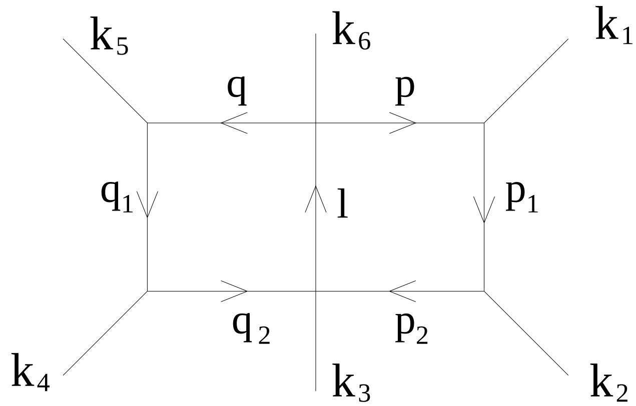

Fig. 5: An arbitrary double-box configuration.

Let us perform, for example, the p-integration. The integral over p,

$$\mathcal{I}_p = \int d^4p\delta(p^2)\delta((p-k_1)^2)\delta((p-k_1-k_2)^2)\delta((p+q+k_6)^2), \tag{4.2}$$

is localized and the answer is [34]

$$\mathcal{I}_p = \frac{2}{(k_1+k_2)^2(k_1+k_6+q)^2\rho}, \tag{4.3}$$

where

$$\begin{aligned} &\rho = \sqrt{1-2(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)^2}, \\ &\lambda_1 = \frac{k_1^2(k_3+k_4+k_5-q)^2}{(k_1+k_2)^2(k_1+k_6+q)^2}, \quad \lambda_2 = \frac{k_2^2(k_6+q)^2}{(k_1+k_2)^2(k_1+k_6+q)^2}. \end{aligned} \tag{4.4}$$

The crucial observation is that when

$$\rho = 1, \tag{4.5}$$

$\mathcal{I}$ acquires an extra propagator-type singularity, i.e. $1/(k_1+k_6+q)^2$. We can formally cut the new propagator by replacing it with a delta-function creating an eighth cut. In other words, after performing the p-integration we end up with following integral over q (we omit the overall q-independent factor)

$$\mathcal{I}_q = \int d^4q\delta(q^2)\delta((q-k_4)^2)\delta((q-k_3-k_4)^2)\frac{1}{(k_1+k_6+q)^2}. \tag{4.6}$$

This integral looks like a triple cut of the following effective box

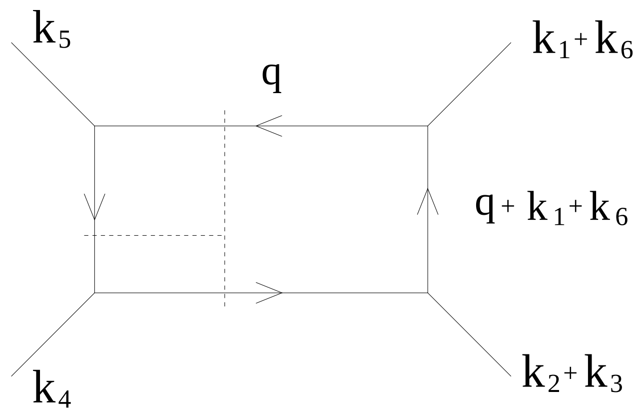

Fig. 6: Effective box that arises after a quadruple cut is used to localize the p integral. The momentum flowing along the uncut line is $q + k_1 + k_6$.

Note that the momentum flowing along the uncut line is exactly $q + k_1 + k_6$. From this viewpoint it is natural to cut the remaining propagator. Note that this procedure localizes the q-integral. Then it is straightforward to write down the coefficients of such double-box integrals. They are given by

$$c_\alpha = -\frac{i}{|\mathcal{S}|} \sum_{h,J_1,J_2,\mathcal{S}} (n_{J_1} n_{J_2} A^{tree}_{(1)} A^{tree}_{(2)} A^{tree}_{(3)} A^{tree}_{(4)} A^{tree}_{(5)} A^{tree}_{(6)})_h, \tag{4.7}$$

where the sum over h is the sum over all helicity configurations, the sums over J_1 and J_2 are the sums over all particles that can propagate in both loops, $\mathcal{S}$ is the set of all solutions for the internal lines of the following system of equations

$$\begin{aligned} &p^2 = 0, \quad (p - k_1)^2 = 0, \quad (p - k_1 - k_2)^2 = 0, \quad (p + q + k_6)^2 = 0, \\ &q^2 = 0, \quad (q - k_5)^2 = 0, \quad (q - k_4 - k_5)^2 = 0, \quad (k_1 + k_6 + q)^2 = 0, \end{aligned} \tag{4.8}$$

and $|\mathcal{S}|$ is the number of solutions. This expression is analogous to the formula for one-loop coefficients of box integrals [34]. It is important to remember that this discussion is valid if

$$\rho = 1, \quad \lambda_1 = \lambda_2 = 0. \tag{4.9}$$

Otherwise, the singularity $1/(k_1 + k_6 + q)^2$ will be replaced by a more complicated one which is not propagator-like, as it can easily be seen from eq. (4.4). The conditions given in (4.9) are satisfied if a given box has two adjacent three-particle vertices. It easy to check that this is always the case if the number of gluons is less than seven. This means that every double-box coefficient of any gluon amplitude with less than seven external lines is given by eq. (4.7). The first double-box configuration where eq. (4.9) is not satisfied appears when the number of external gluons is seven and is shown in fig. 7.

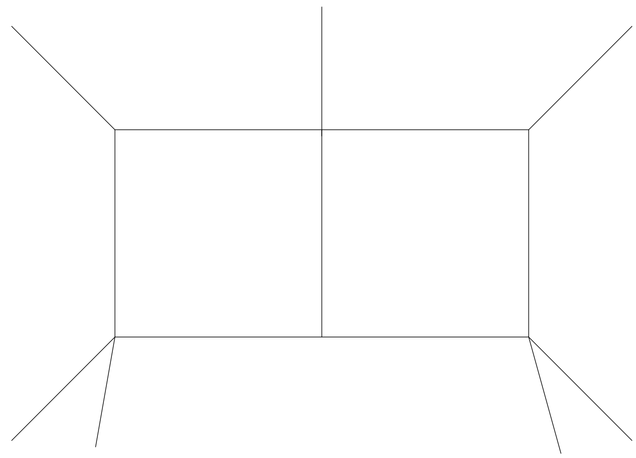

Fig. 7: The simplest double-box configuration for which the conditions in (4.9) are not satisfied.

However, even if the number of external gluons is greater than six, there are double-box configurations for which eq. (4.9) is satisfied. In such cases the eighth cut still exists and eq. (4.7) is still correct.

In fact, eq. (4.7) requires some additional explanations. Note that existence of the effective box in fig. 6 implies that either the momentum l or the momentum p_1 in fig. 5 vanishes. In Minkowski space, this would mean that some tree level amplitudes in eq. (4.7) vanish. Moreover, in Minkowski space, the system of equations (4.8) does not have solutions, which means that we cannot see the singularities under consideration. Therefore, it is not surprising that eq. (4.7), at least naively, is meaningless in Minkowski space. In order to see the new kind of singularities, we have to analytically continue all momenta to signature $(--++)$. But in signature $(--++)$, the statement that a tree amplitude vanishes when one of the incoming or outgoing momentum vanishes is not correct. Each tree amplitude is constructed by using spinors. When one of the incoming or outgoing $(--++)$ momenta vanishes, it is impossible to determine its spinors components even up to rescaling. This leaves the amplitude undetermined. For example, assume that we have a helicity configuration containing a three-gluon amplitude $A(p^-, p_1^-, k_1^+)$. It is given by

$$A(p^-, p_1^-, k_1^+) = \frac{\langle p_1\ p\rangle^3}{\langle p\ k_1\rangle\langle k_1\ p_1\rangle}. \tag{4.10}$$

If p_1 vanishes, the spinor λ_{p_1} cannot be uniquely determined. In fact, λ_{p_1} is not uniquely defined even for non-zero p_1 as it is defined up to rescaling. However, when $p_1 = 0$ the freedom in not being able to determine λ_{p_1} becomes much larger. One can always say

that $p_1 = 0$ implies that $\widetilde{\lambda}_{p_1} = 0$ and λ_{p_1} is arbitrary. This means that $A(p^-, p_1^-, k_1^+)$ becomes arbitrary. Therefore, the numerator in eq. (4.7) is a discontinuous function of momenta and we have to give a prescription on how to define it as l or p_1 goes to zero. The natural way to define it is as follows. Consider first the loop with momentum p. Let $A^{tree}_{(1)}, A^{tree}_{(2)}, A^{tree}_{(3)}$ and $A^{tree}_{(4)}$ be the four tree amplitudes which depend on p. Assuming that they are all non-zero, we can solve the first four p-dependent equations in (4.8) to determine p as a function of the external momenta and q and then evaluate the product $A^{tree}_{(1)} A^{tree}_{(2)} A^{tree}_{(3)} A^{tree}_{(4)}$ on these solutions. We claim that this product can be simplified in such a way that it is a well-defined function when the constraint $(k_1 + k_6 + q)^2 = 0$ is imposed. Below, we will present a few examples that show that this is indeed the case. Having found the product $A^{tree}_{(1)} A^{tree}_{(2)} A^{tree}_{(3)} A^{tree}_{(4)}$ as a function of the external momenta and q, we then multiply it by the remaining two tree amplitudes $A^{tree}_{(5)}$ and $A^{tree}_{(6)}$ and evaluate the product on the solution of the remaining four equations in (4.8). We propose this as a method for calculating double-box coefficients provided conditions (4.9) are fulfilled.

A Subtlety

There is one important subtlety we have to discuss before presenting examples. Consider a helicity configuration with two adjacent three-particle vertices, one of which depends only on internal momenta and the other one depends on both internal and external momenta, with both vertices having the same helicity configuration. For example, consider the configuration in fig. 8.

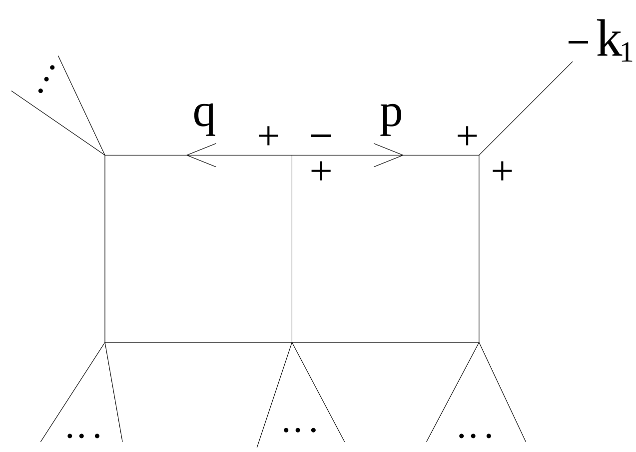

Fig. 8: This helicity configuration is non-zero only if $\lambda_q \sim \lambda_1$.

This configuration is non-zero only if $\lambda_q \sim \lambda_1$. Therefore, the integral over p

$$\int d^4p\delta(p^2)\delta((p-k_1)^2)\delta((p-k_1-k_2)^2)\delta((p+q)^2)A^{tree}_{(1)}A^{tree}_{(2)}A^{tree}_{(3)}A^{tree}_{(4)} \tag{4.11}$$

must be proportional to $\delta((k_1+q)^2)$. In other words, the integral lacks the extra propagator-like singularity and therefore it does not contribute to the octa-cut.

4.2. Split Double-Box Scalar Integrals

When the number of gluons is greater than five, a new kind of double box integrals can appear. These were introduced in the introduction in fig. 1b. For the reader's convenience we depict them again in fig. 9. This double box scalar integrals are such that the two boxes only share a vertex and not a propagator. This is why we will call them split double boxes.

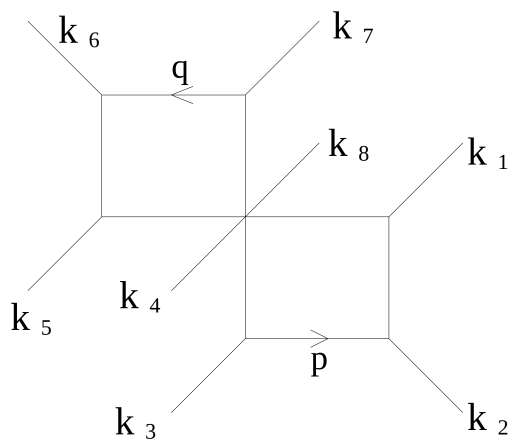

Fig. 9: Generic split double box configuration.

The coefficients of the split double boxes are easy to compute. Since they have eight propagators and the two loop integrations are completely independent, it is straightforward to consider two quadruple cuts or equivalently an honest octa-cut. This produces eight delta-functions that localize both loop integrals. Let us see this in more detail. The quadruple cut in the q-loop fixes q to be a solution to the following equations,

$$q^2=0,\ (q-k_6)^2=0,\ (q-k_5-k_6)^2=0,\ (q+k_7)^2=0, \tag{4.12}$$

while a quadruple cut in the p-loop fixes p to be a solution of

$$p^2=0,\ (p-k_2)^2=0,\ (p-k_1-k_2)^2=0,\ (p+k_3)^2=0. \tag{4.13}$$

Each set of equations has two solutions. The coefficient of a split double-box scalar integral is then given by

$$c = \frac{1}{4} \sum_{h, J_1, J_2, \mathcal{S}} (n_{J_1} n_{J_2} A_{(1)}^{tree} A_{(2)}^{tree} A_{(3)}^{tree} A_{(4)}^{tree} A_{(5)}^{tree} A_{(6)}^{tree} A_{(7)}^{tree})_h \tag{4.14}$$

where we have used that the number of solutions is 4.

5. Examples

In this section we consider several examples of coefficients calculated by using octa-cuts. All of them are coefficients of scalar double boxes with seven propagators. We consider four-, five- and six-gluon amplitudes. For six-gluons we study a non-MHV amplitude with adjacent negative helicity gluons.

5.1. Four-Gluon Amplitude $A^{2-loop}(-,-,+,+)$ Revisited

As a first example, let us reconsider the octa-cut of the four-gluon amplitude from section 3 in the s-channel. The octa cut in the t-channel is analogous. The additional propagator that we cut is $\frac{1}{(q+K_1)^2}$. See fig. 3 for notation. Taking into account the subtlety in the previous section, there are only four helicity configurations that contribute. All of them give the same answer $-A_4^{tree} s^2 t$. On the other hand, the number of solutions to eqs. (4.8) can be shown to be four. This gives $c_s = -A_4^{tree} s^2 t$ as in eq. (3.8). Note that the product of the four tree level amplitudes depending on the internal momentum p is given by $\langle 1\ 2\rangle^2 [q\ q_2]^2$ (see eq. (3.11)). This expression does not have any ambiguity in the presence of the eighth delta-function $\delta((q+K_1)^2)$.

5.2. Five-Gluon Amplitude $A^{2-loop}(-,-,+,+,+)$

As a next example, let us calculate the coefficient of the following five-gluon double-box configuration.

In this case, there are two helicity configurations to consider. Both of them can be shown to give the same contribution. We will consider the helicity configuration shown in fig. 11.

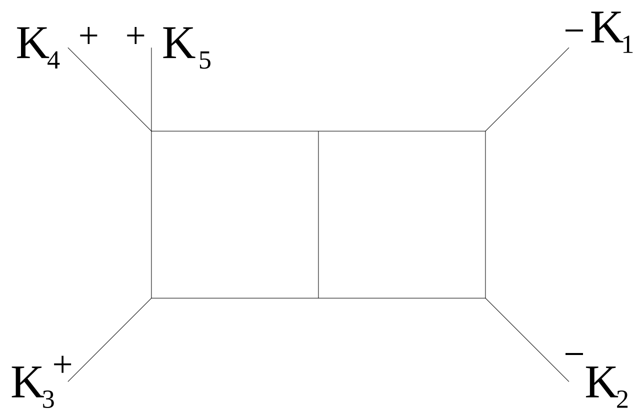

Fig. 10: A double box integral of the five-gluon amplitude $A(1^-,2^-,3^+,4^+,5^+)$.

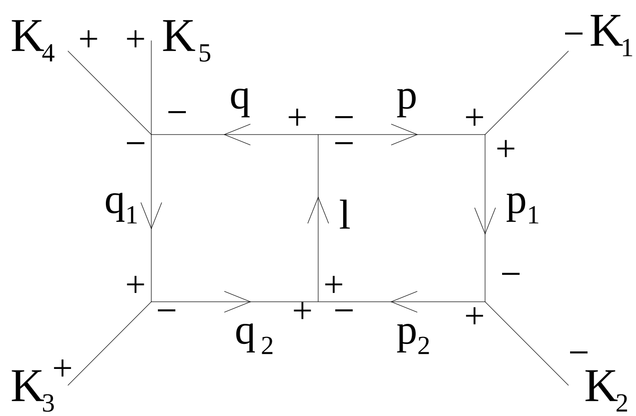

Fig. 11: One of the two possible helicity configurations contributing to the coefficient of the integral of fig. 10.

Note that only gluons can propagate in both loops. The product of the six tree level amplitudes is as follows

$$\langle 1\ 2\rangle^2 [q\ q_2]^2 \frac{[3\ q_1]^3}{[q_1\ q_2][q_2\ 3]} \frac{\langle q\ q_1\rangle^3}{\langle q_1\ 4\rangle\langle 4\ 5\rangle\langle 5\ q\rangle}, \tag{5.1}$$

where the first two factors come from the four vertices on the right. The computation leading to the first two factors was done in the previous section in eq. (3.11). By using momentum conservation along the lines of eq. (3.10), eq. (5.1) can be reduced to

$$\frac{\langle 1\ 2\rangle^2 s^2}{\langle 3\ 4\rangle\langle 4\ 5\rangle} \left([5\ 3] + [4\ 3] \frac{\langle q\ 4\rangle}{\langle q\ 5\rangle} \right), \tag{5.2}$$

Now we impose the delta-function $\delta((K_1+q)^2)$. It has two possible solutions, $\lambda_q \sim \lambda_1$ or $\widetilde{\lambda}_q \sim \widetilde{\lambda}_1$. It is not hard to show that if we choose $\widetilde{\lambda}_q \sim \widetilde{\lambda}_1$ then the expression in eq. (5.2)

vanishes. Therefore, the only relevant solution is $\lambda_q \sim \lambda_1$. Then the octa-cut becomes

$$\frac{\langle 1\ 2\rangle^2 s^2}{\langle 3\ 4\rangle\langle 4\ 5\rangle}\left([5\ 3]+[4\ 3]\frac{\langle 1\ 4\rangle}{\langle 1\ 5\rangle}\right)\int d\mu. \tag{5.3}$$

Taking into account that the system of equations (4.8) has four solutions, we find that the corresponding coefficient is

$$c_1^{(5)} = -\frac{2i}{4}\frac{\langle 1\ 2\rangle^2 s^2}{\langle 3\ 4\rangle\langle 4\ 5\rangle}\left([5\ 3]+[4\ 3]\frac{\langle 1\ 4\rangle}{\langle 1\ 5\rangle}\right) = -\frac{1}{2}A_5^{tree}s^2 t, \tag{5.4}$$

where $s=(K_1+K_2)^2$ and $t=(K_2+K_3)^2$ and A_5^{tree} is the tree-level five-gluon amplitude

$$A_5^{tree} = i\frac{\langle 1\ 2\rangle^3}{\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 1\rangle}. \tag{5.5}$$

Let us consider one more example. Let us calculate the coefficient of the five-gluon double-box configuration shown in fig. 12.

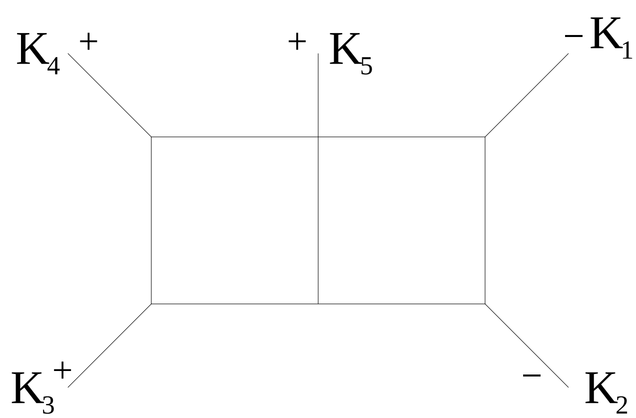

Fig. 12: Second example of a double-box configuration in the five-gluon amplitude $A(1^-,2^-,3^+,4^+,5^+)$.

There are two helicity configurations to consider. However, one of them gives the zero answer. The non-zero contribution comes from the helicity configuration shown in fig. 13.

Note that only gluons can propagate in both loops. The number of solutions to eqs. (4.8) can be shown to be two. Then the corresponding coefficient is given by

$$\begin{aligned} c_2^{(5)} = &-\frac{i}{2}\frac{\langle q_1\ q_2\rangle^3}{\langle q_2\ 3\rangle\langle 3\ q_1\rangle}\frac{[q_1\ 4]^3}{[4\ q][q\ q_1]}\frac{\langle p\ l\rangle^3}{\langle l\ q\rangle\langle q\ 5\rangle\langle 5\ p\rangle}\frac{[q_2\ l]^3}{[l\ p_2][p_2\ q_2]} \\ &\frac{\langle p_1\ 2\rangle^3}{\langle 2\ p_2\rangle\langle p_2\ p_1\rangle}\frac{[p_1\ p]^3}{[p\ 1][2\ p_1]}. \end{aligned} \tag{5.6}$$

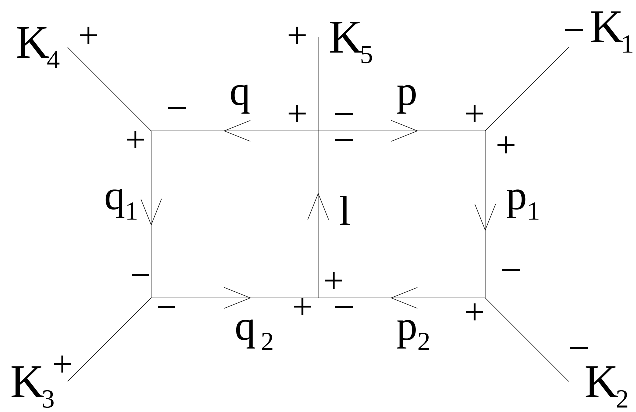

Fig. 13: The only non-vanishing helicity configuration contributing to the coefficient of the double box integral of fig. 12.

Using momentum conservation and the fact that $\lambda_q \sim \lambda_4$, eq (5.6) can be simplified to give

$$c_2^{(5)} = -\frac{1}{2} A_5^{tree} stu, \tag{5.7}$$

where $u = (K_3 + K_4)^2$. All other double-box coefficients of the five-gluon amplitude can be found by analogous calculations.

5.3. Six-Gluon Amplitude $A^{2-loop}(-,-,-,+,+,+)$

As our next example, let us calculate the coefficient of the six-gluon next-to-MHV double-box configuration shown in fig. 14. The additional singularity that we cut is again $\frac{1}{(q+K_1)^2}$. There are two helicity configurations to consider, both giving the same answer. In both of them only gluons can propagate in both loops. Let us describe the calculation of the one depicted in fig. 15.

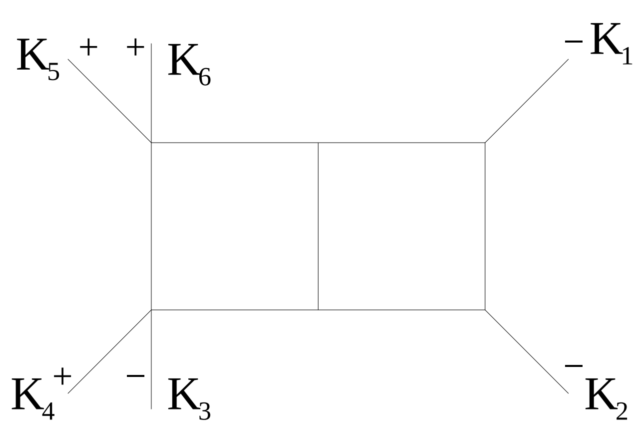

Fig. 14: A double-box integral of the six-gluon non-MHV amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$.

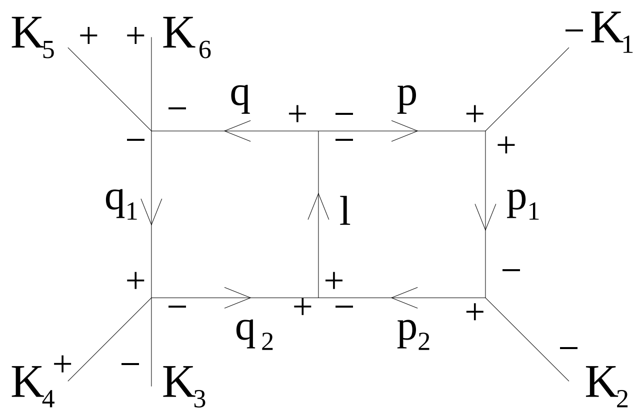

Fig. 15: One of the two helicity configurations contributing to the coefficient of the scalar double box of fig.14.

The product of the four p-dependent tree amplitudes gives (see eq. (3.11))

$$\langle 1\ 2\rangle^2 [q\ q_2]^2. \tag{5.8}$$

Then the numerator of eq. (4.7) becomes

$$\begin{gathered} -2i(A_{(1)}^{tree} A_{(2)}^{tree} A_{(3)}^{tree} A_{(4)}^{tree} A_{(5)}^{tree} A_{(6)}^{tree}) = \\ \langle 1\ 2\rangle^2 [q\ q_2]^2 \frac{\langle q_2\ 3\rangle^3}{\langle 3\ 4\rangle\langle 4\ q_1\rangle\langle q_1\ q_2\rangle} \frac{[5\ 6]^3}{[6\ q][q\ q_1][q_1\ 5]}, \end{gathered} \tag{5.9}$$

where the factor of two comes from two helicity configurations. Using momentum conservation similar to eq. (3.11), eq. (5.9) can be simplified as follows

$$i \frac{\langle 1\ 2\rangle^2 [5\ 6]^3 \langle 3|5+6|q]^2 \langle q_2\ 3\rangle}{\langle 3\ 4\rangle [6\ q] \langle 4|5+6|q] \langle q_2|3+4|5]}. \tag{5.10}$$

Now we impose the last condition $(q+K_1)^2 = 0$. This equation has two solutions. We can either have $\widetilde{\lambda}_q \sim \widetilde{\lambda}_1$ or $\lambda_q \sim \lambda_1$. Both solutions give non-zero contributions. The first solution yields

$$i \frac{\langle 1\ 2\rangle^2 [5\ 6]^3 \langle 3|5+6|1]^2 \langle 2\ 3\rangle}{\langle 3\ 4\rangle [6\ 1] \langle 4|5+6|1] \langle 2|3+4|5]} \tag{5.11}$$

while the second solution yields

$$i \frac{\langle 1\ 2\rangle^2 [5\ 6]^3 \langle 5\ 6\rangle [4\ 2]^2 \langle 3|(1+2)\cdot(5+6)\cdot(3+4)|2]}{[2\ 3] \langle 5|6+1|2] [5|(3+4)\cdot(1+2)\cdot(5+6)\cdot(3+4)|2]}. \tag{5.12}$$

Taking into account that the system (4.8) in this case has four solutions, the double-box coefficient becomes

$$c_1^{(6)} = \frac{i}{2}\left(\frac{\langle 1\ 2\rangle^2[5\ 6]^3\langle 3|5+6|1]^2\langle 2\ 3\rangle}{\langle 3\ 4\rangle[6\ 1]\langle 4|5+6|1]\langle 2|3+4|5]} + \right.$$
$$\left.\frac{\langle 1\ 2\rangle^2[5\ 6]^3\langle 5\ 6\rangle[4\ 2]^2\langle 3|(1+2)\cdot(5+6)\cdot(3+4)|2]}{[2\ 3]\langle 5|6+1|2][5|(3+4)\cdot(1+2)\cdot(5+6)\cdot(3+4)|2]}\right). \tag{5.13}$$

Let us consider one more example. Let us calculate the coefficient of the six-gluon double-box configuration shown in fig. 16.

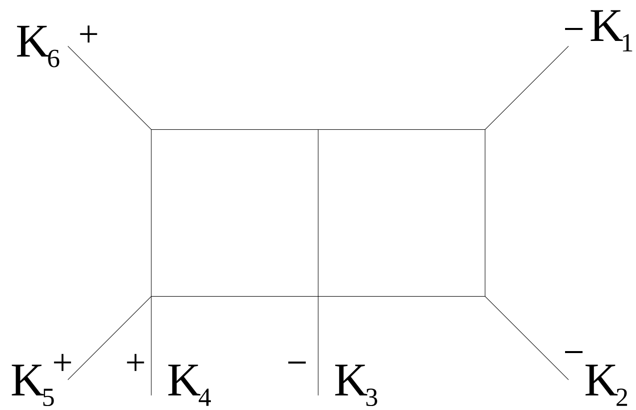

Fig. 16: Second example of a six-gluon double-box integral of the six-gluon non-MHV amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$.

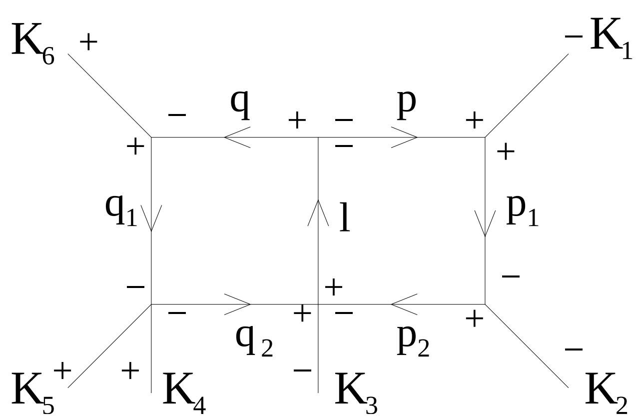

Fig. 17: The only non-vanishing helicity configuration contributing to the coefficient of the scalar double-box integral of fig.16.

In this case, there is only one helicity configuration contributing to the octa-cut. It is shown in fig. 17. Only gluons can propagate in both loops. The system of equations (4.8)

has two solutions. Therefore, the corresponding double-box coefficient is given by

$$c_2^{(6)} = -\frac{i}{2} \frac{[p_1\ p]^3}{[p_1\ 1][1\ p_1]} \frac{\langle p_1\ 2\rangle^3}{\langle 2\ p_2\rangle\langle p_2\ p_1\rangle} \frac{\langle p\ l\rangle^3}{\langle l\ q\rangle\langle q\ p\rangle} \frac{[q_2\ l]^3}{[l\ p_2][p_2\ 3][3\ q_2]} \times \\ \frac{[q_1\ 6]^3}{[6\ q][q\ q_1]} \frac{\langle q_1\ q_2\rangle^3}{\langle q_2\ 4\rangle\langle 4\ 5\rangle\langle 5\ q_1\rangle}. \tag{5.14}$$

By using the first seven equations in (4.8), we can simplify (5.14) as follows

$$c_2^{(6)} = -\frac{i}{2} \frac{u^3 s\langle 1|q|6]}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 4|q - K_4^{[3]}|3]}, \tag{5.15}$$

where

$$u = (K_4 + K_5 + K_6)^2, \quad s = (K_1 + K_2)^2. \tag{5.16}$$

Now we consider the last equation $(q + K_1)^2 = 0$. From fig. 17, it follows that λ_q has to be proportional to λ_6. Therefore, $\widetilde{\lambda}_q$ has to be proportional to $\widetilde{\lambda}_1$. Using momentum conservation, one can find that

$$q = -\frac{u}{\langle 6|4+5|1]} \lambda_6 \widetilde{\lambda}_1. \tag{5.17}$$

Substituting this into eq. (5.16), we obtain

$$c_2^{(6)} = -\frac{i}{2} \frac{u^4 st}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 4|5+6|1]\langle 6|4+5|3]}, \tag{5.18}$$

where

$$t = (K_1 + K_6)^2. \tag{5.19}$$

All other double-box coefficients can be computed by similar calculations.

The calculation of the coefficient $c_2^{(6)}$ can be generalized for the configuration considered in fig. 18.

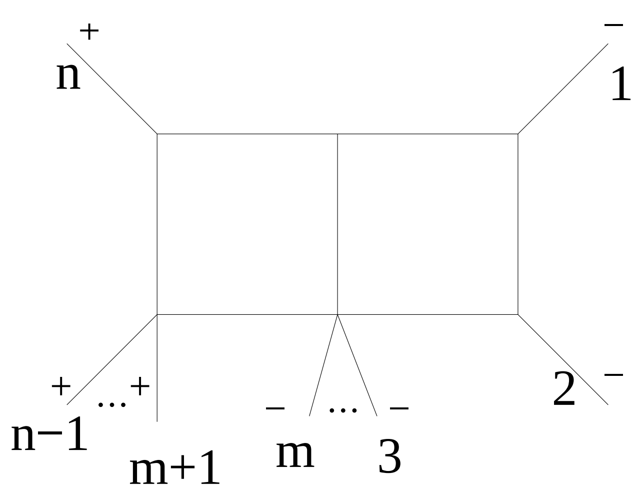

Fig. 18: An infinite family of n-gluon double box scalar integrals.

There is only one helicity configuration that contributes to the octa-cut. A similar calculation gives

$$c^{(n)} = -\frac{i}{2}\frac{u^4 st}{[1\ 2][2\ 3]\ldots[m-1\ m]\langle m+1\ m+2\rangle\ldots\langle n-1\ n\rangle}\times \\ \frac{1}{\langle m+1|K_{m+1,m+2,\ldots,n}|1]\langle n|K_{m+1,m+2,\ldots,n}|m]}, \tag{5.20}$$

where s and t are given by eq. (5.16) and u is given by

$$u = (K_{m+1,m+2,\ldots,n})^2 = (K_{m+1} + K_{m+2} + \ldots + K_n)^2. \tag{5.21}$$

6. Application to Three and Higher Loops

Ideas presented in the previous sections can be applied to higher loops. Let us consider triple-box configurations appearing at three-loops. The configurations we consider are obtained from the double-box configurations at two loops by adding three new propagators to form the third loop. This way, one can produce a ladder diagram as in fig. 19a or a double box with a pentagon as in fig. 19b. We make a slight abuse of terminology and call both kind of configurations triple-box diagrams.

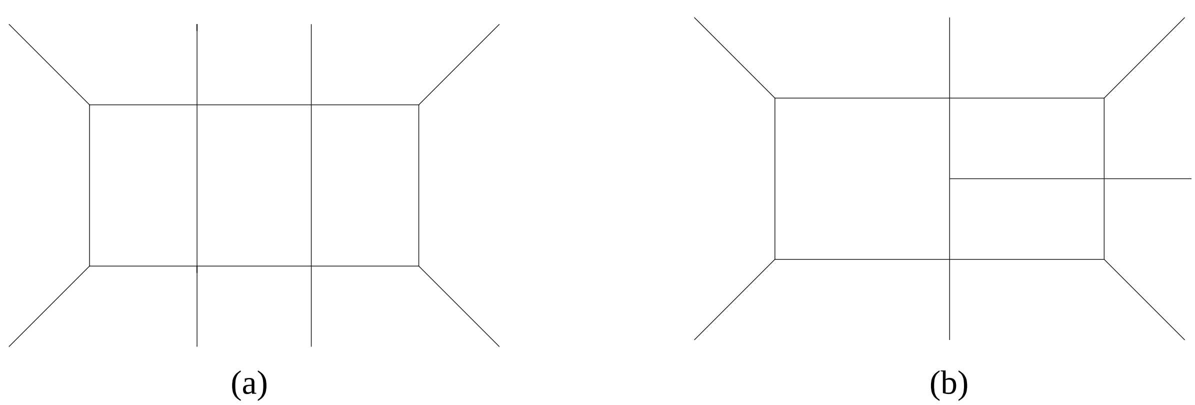

Fig. 19: Three-loop triple box configurations. (a) A triple box ladder integral. (b) A double box with a pentagon.

Every triple box contains ten propagators[7]. Therefore it is natural to start with a ten-particle cut. This produces ten delta-functions whereas the number of integration variables

[7] Of course, for a large enough number of gluons one can also find split triple boxes which can have 11 or 12 propagators.

is twelve. However, it follows from our previous analysis that box configurations develop additional propagator-like singularities which can also be cut (replaced by their discontinuities). A triple-box configuration naturally admits two extra propagator-type singularities which allows us to consider twelve-particle cuts. Therefore, it should be possible to completely localize all momentum integrals, at least if the number of gluons is not big enough. Then it is straightforward to write down an expression for the coefficients analogous to eq. (4.7). Obviously, a similar analysis can be performed at any number of loops. It is interesting to mention that at three loops, some of the triple boxes that enter in the calculation of the four-gluon amplitude are not scalar triple boxes [44]. This means that the numerator in the integrand is not one but an inverse propagator. See fig. 20. This was also found to be the case for higher loops [44].

We start our discussion with the analysis of ladder diagrams which allow a straightforward generalization of our discussion in section 4. Then we turn to the triple-box integrals where one of the "boxes" has five propagators and see how our analysis of singularities realizes the phenomenon mentioned above. For concreteness, we will concentrate on the four-gluon amplitude though an identical analysis can be performed regardless of the number of external lines. Let us start with the triple-box configuration in fig. 21.

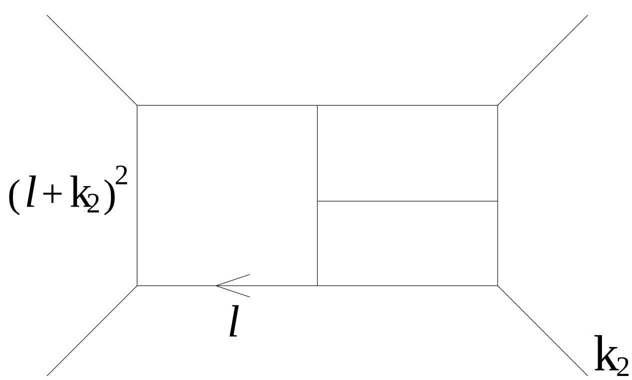

Fig. 20: Schematic representation of a modified "triple box" integral given in [44].

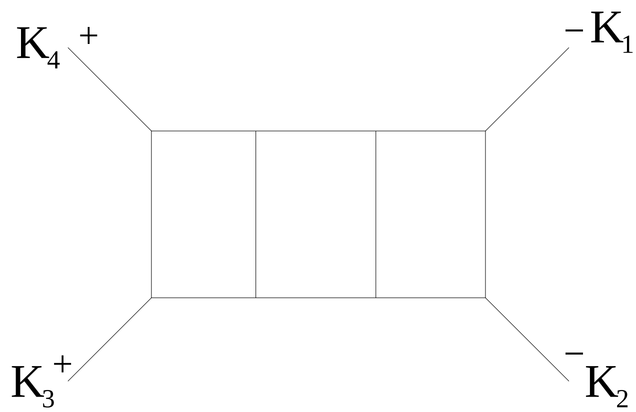

Fig. 21: A ladder triple-box configuration with four external gluons.

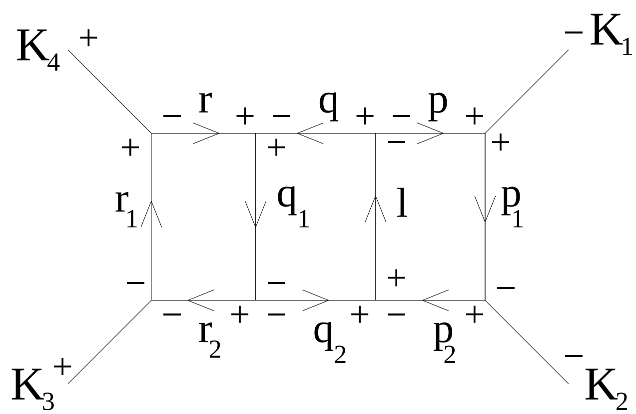

Fig. 22: One of the helicity configurations contributing to the coefficient of the ladder triple-box in fig.21.

In this case, there are twelve helicity configurations, all giving the same answer. It is enough to consider one of them, for example the one in fig. 22. Similarly to the two-loop case, it is enough to consider the ten-particle cut because the measure integral factors out and cancels. Then the corresponding coefficient is given by

$$d_1 = -i^3 \widetilde{A}_4^{tree} s^2 \widetilde{t}^2 \frac{\langle r_1\ r_2\rangle^3}{\langle r_2\ 3\rangle\langle 3\ r_1\rangle} \frac{[r_1\ 4]^3}{[4\ r][r\ r_1]}, \tag{6.1}$$

where $\widetilde{A}_4^{tree}$ is the tree level four-gluon amplitude with the external lines $(K_1^-, K_2^-, r_2^+, r_1^+)$, $s = (K_1 + K_2)^2$ and $\widetilde{t} = (K_1 - r)^2$. In eq. (6.1), we used that the product of the six tree amplitudes was computed in section 3. Using momentum conservation, eq. (6.1) can be simplified to give

$$d_1 = -iA_4^{tree} s^3 t. \tag{6.2}$$

where $t = (K_2 + K_3)^2$. This coincides with the answer from [44].

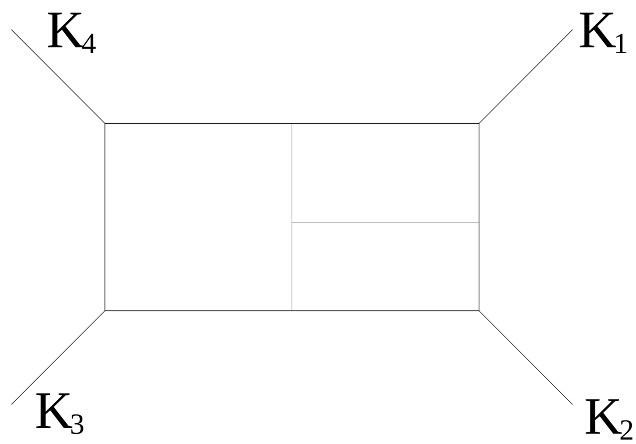

Fig. 23: A triple-box configuration.

Now let us consider the configuration in fig. 23. Note that one of the loops has five propagator and this is why we said that the configuration was a double box with a pentagon. This is the basic reason why the integral is not a scalar box integral as we will see below. Let us start our analysis with the measure integral

$$\begin{aligned}\mathcal{I}_0 = \int d^4l d^4q d^4p \delta(l^2)\delta((l-K_3)^2)\delta((l-K_3-K_4)^2)\delta((l-K_3-K_4-K_1)^2) \\ \delta((p-K_2-l)^2)\delta((p-K_2)^2)\delta((q-K_1)^2)\delta((p+q)^2)\delta(p^2)\delta(q^2)\end{aligned} \tag{6.3}$$

and perform the integration over p and q. After integrating over p we obtain (up to the momenta-independent factor)

$$\frac{1}{(K_2+q)^2(K_2+l)^2}. \tag{6.4}$$

Then we cut the "propagator" $1/(K_2+q)^2$. This gives the fourth delta-function which allows us to perform the integration over q. This produces (again, we ignore the momenta-independent factors) one more factor of $(K_2+l)^2$ in the denominator. Thus, we obtain that this triple box configuration has a singularity

$$\left(\frac{1}{(K_2+l)^2}\right)^2. \tag{6.5}$$

Let us calculate the coefficient of this triple box integral by calculating the product of the eight-gluon amplitudes. For every non-vanishing helicity configuration, the product of the six gluon amplitudes on the right is the coefficient c_t of the two-loop four-gluon amplitude studied in section 3 and it is given by

$$-A_4^{\prime tree} st^{\prime 2}, \tag{6.6}$$

where $A_4'^{tree}$ is the tree four-gluon amplitude with external lines $(K_1^-, K_2^-, l^+, (l - K_3 - K_4)^+)$ and t' is given by

$$t' = (K_2 + l)^2. \tag{6.7}$$

Therefore, in attempting to calculate the integral of the product of the tree amplitudes, the singularity (6.5) cancels out. This means that the coefficient of the triple box in fig. 23 is zero. This is in agreement with results of [44]. On the other hand, the amplitude $A_4'^{tree}$ has a factor

$$\frac{1}{\langle 2\ l \rangle} = \frac{[2\ l]}{(K_2 + l)^2}. \tag{6.8}$$

This indicates that the actual diagram has a singularity $\frac{1}{(K_2+l)^2}$. In order to account for this singularity we have to introduce a slightly modified triple box integral schematically shown in fig. 20. The basic idea is to multiply in the numerator by $(K_2 + l)^2$ in order to cancel one of the power in (6.5) and get the correct $1/(K_2 + l)^2$ singular behavior. This shows that this triple-box integral should be in the list of scalar integrals of the amplitude under study. This is completely consistent with results of [44]. Now we can cut this "propagator" and completely localize the integral. Note that the combination

$$[2\ l]\delta((K_2 + l)^2) \tag{6.9}$$

is not necessarily zero. This just means that we have to choose the solution

$$\lambda_l \sim \lambda_2. \tag{6.10}$$

The coefficient of this modified box is now straightforward to compute. The answer is

$$d_2 = -iA_4^{tree} s^2 t. \tag{6.11}$$

This coincides with the corresponding coefficient from [44]. All other coefficients of this amplitude can be found in a similar manner. Even though we concentrated on the four-gluon amplitude, we could do the same analysis for any amplitude admitting additional cuts.

As mentioned in the introduction, the basis of integral for $\mathcal{N} = 4$ L-loop amplitudes of gluons is not known except for $L = 1$. One can imagine that a more systematic analysis along the lines of the discussion presented in this section might give a way of obtaining such a basis. It would be interesting to explore this direction in the future.

7. Conclusion

In this paper, we observed that certain scalar double-box integrals which appear at two loops in $\mathcal{N} = 4$ Yang-Mills theory possess hidden singularities. Such singularities are manifest after a quadruple cut is performed on one of the boxes. The end result is that one can straightforwardly calculate the coefficient of such integrals by an octa-cut which localizes the cut integral. The form of the coefficient is universal and it is given by the product of a certain number of tree-level amplitudes. This technique is applicable to all scalar double box integrals in amplitudes with less than seven external gluons and to a large subset of double box integrals for seven or more external gluons. The basis of integrals at two loops is not known in general. For four gluons the amplitude is given in terms of only scalar double-box integrals. If it turns out that the basis of integrals for five- and six-gluon amplitudes is also given by scalar double box integrals, then our technique gives a simple way of computing all those amplitudes for any helicity configuration. We also argued that this technique can be applied to higher loop amplitudes. At three loops we found that our technique can be easily extended to compute the coefficient of ladder diagrams. For the class of diagrams with a pentagon, our method shows that the coefficient of scalar integrals is zero and naturally gives the modified integral for which the coefficient does not vanish.

Acknowledgments

The authors would like to thank Z. Bern, R. Britto, L. Dixon, B. Feng, H. Osborn, M. Spradlin and A. Volovich for comments on the first version of the manuscript. Research of E.I.B is supported by NSF grant PHY-0070928. Research of F.C. is supported in part by the Martin A. and Helen Chooljian Membership at the Institute for Advanced Study and by DOE grant DE-FG02-90ER40542. Research at the Perimeter Institute is supported by funds from NSERC of Canada.

References

[1] E. Witten, "Perturbative gauge theory as a string theory in twistor space", Commun. Math. Phys. 252, 189 (2004) [hep-th/0312171].

[2] F. Cachazo, P. Svrček and E. Witten, "MHV vertices and tree amplitudes in gauge theory", JHEP 0409:006 (2004) [hep-th/0403047].

[3] C. J. Zhu, "The googly amplitudes in gauge theory", JHEP 0404:032 (2004) [hep-th/0403115].

[4] G. Georgiou and V. V. Khoze, "Tree amplitudes in gauge theory as scalar MHV diagrams", JHEP 0405:070 (2004) [hep-th/0404072].

[5] J. B. Wu and C. J. Zhu, "MHV vertices and scattering amplitudes in gauge theory", JHEP 0407:032 (2004) [hep-th/0406085].

[6] J. B. Wu and C. J. Zhu, "MHV vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos", JHEP 0409:063 (2004) [hep-th/0406146].

[7] D. A. Kosower, "Next-to-maximal helicity violating amplitudes in gauge theory", Phys. Rev. D71:045007 (2005) [hep-th/0406175].

[8] G. Georgiou, E. W. N. Glover and V. V. Khoze, "Non-MHV tree amplitudes in gauge theory", JHEP 0407:048 (2004) [hep-th/0407027].

[9] Y. Abe, V. P. Nair and M. I. Park, "Multigluon amplitudes, N = 4 constraints and the WZW model", Phys. Rev. D71:025002 (2005) [hep-th/0408191].

[10] L. J. Dixon, E. W. N. Glover and V. V. Khoze, "MHV rules for Higgs plus multi-gluon amplitudes", JHEP 0412:015 (2004) [hep-th/0411092].

[11] S. D. Badger, E. W. N. Glover and V. V. Khoze, "MHV rules for Higgs plus multi-parton amplitudes", JHEP 0503:023 (2005) [hep-th/0412275].

[12] Z. Bern, D. Forde, D. A. Kosower and P. Mastrolia, "Twistor-inspired construction of electroweak vector boson currents", hep-ph/0412167.

[13] R. Roiban, M. Spradlin and A. Volovich, "Dissolving N = 4 loop amplitudes into QCD tree amplitudes", Phys. Rev. Lett. 94:102002 (2005) [hep-th/0412265].

[14] R. Britto, F. Cachazo and B. Feng, "New recursion relations for tree amplitudes of gluons", Nucl. Phys. B715 (2005) 499-522 [hep-th/0412308].

[15] R. Britto, F. Cachazo, B. Feng and E. Witten, "Direct proof of tree-level recursion relation in Yang-Mills theory", hep-th/0501052.

[16] M. Luo and C. Wen, "Recursion relations for tree amplitudes in super gauge theories", JHEP 0503:004 (2005) [hep-th/0501121].

[17] M. Luo and C. Wen, "Compact formulas for all tree amplitudes of six partons", hep-th/0502009.

[18] R. Britto, B. Feng, R. Roiban, M. Spradlin and A. Volovich, "All split helicity tree-level gluon amplitudes", hep-th/0503198.

[19] S. D. Badger, E. W. N. Glover, V. V. Khoze and P. Svrček, "Recursion Relations for Gauge Theory Amplitudes with Massive Particles", hep-th/0504159.

[20] Z. Bern, L. J. Dixon and D. A. Kosower, "On-Shell Recurrence Relations for One-Loop QCD Amplitudes", hep-th/0501240.

[21] Z. Bern, L. J. Dixon and D. A. Kosower, "The Last of the Finite Loop Amplitudes in QCD", hep-ph/0505055.

[22] A. Brandhuber, S. McNamara, B. Spence and G. Travaglini, "Loop Amplitudes in Pure Yang-Mills from Generalised Unitarity," hep-th/0506068.

[23] C. Quigley and M. Rozali, "One-loop MHV amplitudes in supersymmetric gauge theories", JHEP 0501:053 (2005) [hep-th/0410278].

[24] J. Bedford, A. Brandhuber, B. Spence and G. Travaglini, "A twistor approach to one-loop amplitudes in N = 1 supersymmetric Yang-Mills theory", Nucl. Phys. B706:100 (2005) [hep-th/0410280].

[25] J. Bedford, A. Brandhuber, B. Spence and G. Travaglini, "Non-Supersymmetric Loop Amplitudes and MHV Vertices", Nucl. Phys. B712:59 (2005) [hep-th/0412108].

[26] S. J. Bidder, N. E. J. Bjerrum-Bohr, L. J. Dixon and D. C. Dunbar, "N = 1 supersymmetric one-loop amplitudes and the holomorphic anomaly of unitarity cuts", Phys. Lett. B606:189 (2005) [hep-th/0410296].

[27] S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, "Twistor space structure of the box coefficients of N = 1 one-loop amplitudes", Phys. Lett. B608:151 (2005) [hep-th/0412023].

[28] S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, "One-loop gluon scattering amplitudes in theories with $N < 4$ supersymmetries", Phys. Lett. B612:75 (2005) [hep-th/0502028].

[29] R. Britto, E. I. Buchbinder, F. Cachazo and B. Feng, "One-loop amplitudes of gluons in SQCD", hep-ph/0503132.

[30] A. Brandhuber, B. Spence and G. Travaglini, "One-loop gauge theory amplitudes in N = 4 super Yang-Mills from MHV vertices", Nucl. Phys. B706:150 (2005) [hep-th/0407214].

[31] F. Cachazo, "Holomorphic Anomaly Of Unitarity Cuts And One-Loop Gauge Theory Amplitudes", hep-th/0410077.

[32] R. Britto, F. Cachazo and B. Feng, "Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts", Phys. Rev. D71:025012 (2005) [hep-th/0410179].

[33] Z. Bern, V. Del Duca, L. J. Dixon and D. A. Kosower, "All non-maximally-helicity-violating one-loop seven-gluon amplitudes in N = 4 super-Yang-Mills theory", Phys. Rev. D71:045006 (2005) [hep-th/0410224].

[34] R. Britto, F. Cachazo and B. Feng, "Generalized Unitarity and One-Loop Amplitudes in N=4 Super-Yang-Mills", hep-th/0412103.

[35] Z. Bern, L. J. Dixon and D. A. Kosower, “All next-to-maximally helicity-violating one-loop gluon amplitudes in N = 4 super-Yang-Mills theory”, hep-th/0412210.

[36] L. J. Dixon, “Calculating Scattering Amplitudes Efficiently,” hep-ph/9601359.

[37] C. Anastasiou, Z. Bern, L. Dixon and D. A. Kosower, “Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory”, Phys.Rev.Lett. 91 (2003) 251602 [hep-th/0309040].

[38] Z. Bern, L. J. Dixon and D. A. Kosower, “Two-Loop $g \to gg$ Splitting Amplitudes in QCD”, JHEP 0408 (2004) 012 [hep-ph/0404293].

[39] D. A. Kosower, “All-Order Collinear Behavior in Gauge Theories”, Nucl.Phys. B552 (1999) 319-336 [hep-ph/9901201].

[40] C. Anastasiou, L. J. Dixon, Z. Bern and D. A. Kosower, “Cross-Order Relations in $N = 4$ Supersymmetric Gauge Theory,” hep-th/0402053

[41] Z. Bern, L. J. Dixon and V. A. Smirnov, “Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond,” hep-th/0505205

[42] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, “One Loop N Point Gauge Theory Amplitudes, Unitarity And Collinear Limits,” Nucl. Phys. B425, 217 (1994) [hep-ph/9403226].

[43] L. M. Brown and R. P. Feynman, “Radiative Corrections To Compton Scattering,” Phys. Rev. 85:231 (1952); G. Passarino and M. Veltman, “One Loop Corrections For E+ E- Annihilation Into Mu+ Mu- In The Weinberg Model,” Nucl. Phys. B160:151 (1979); G. ’t Hooft and M. Veltman, “Scalar One Loop Integrals,” Nucl. Phys. B153:365 (1979); R. G. Stuart, “Algebraic Reduction Of One Loop Feynman Diagrams To Scalar Integrals,” Comp. Phys. Comm. 48:367 (1988); R. G. Stuart and A. Gongora, “Algebraic Reduction Of One Loop Feynman Diagrams To Scalar Integrals. 2,” Comp. Phys. Comm. 56:337 (1990).

[44] Z. Bern, J.S. Rozowsky and B. Yan, “Two-Loop Four-Gluon Amplitudes in N=4 Super-Yang-Mills”, Phys.Lett. B401 (1997) 273-282 [hep-ph/9702424].

[45] R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, *The Analytic S-Matrix*, Cambridge University Press, 1966.

[46] V. A. Smirnov, Phys. Lett. B **460**, 397 (1999) [hep-ph/9905323]; V. A. Smirnov, “Analytical Result for Dimensionally Regularized Massless Master Double Box With One Leg Off Shell,” Phys. Lett. B **491**, 130 (2000) [arXiv:hep-ph/0007032];V. A. Smirnov, “Analytical Result for Dimensionally Regularized Massless Master Non-planar Double Box With One Leg Off Shell,” Phys. Lett. B **500**, 330 (2001) [arXiv:hep-ph/0011056].

[47] F. A. Berends, W. T. Giele and H. Kuijf, “On Relations Between Multi-Gluon And Multi-Graviton Scattering,” Phys. Lett B211 (1988) 91.

[48] Z. Xu, D.-H. Zhang and L. Chang, “Helicity Amplitudes For Multiple Bremsstrahlung In Massless Nonabelian Theories,” Nucl. Phys. **B291** (1987) 392.

[49] J. F. Gunion and Z. Kunszt, "Improved Analytic Techniques For Tree Graph Calculations And The G G Q Anti-Q Lepton Anti-Lepton Subprocess", Phys. Lett. 161B (1985) 333.

[50] Z. Bern and D. A. Kosower, "Color Decomposition Of One Loop Amplitudes In Gauge Theories," Nucl. Phys. B362, 389 (1991)

[51] F. A. Berends and W. Giele, "The Six Gluon Process As An Example Of Weyl-Van Der Waerden Spinor Calculus," Nucl. Phys. B294, 700 (1987)

[52] M. Mangano, S. J. Parke and Z. Xu, "Duality And Multi - Gluon Scattering," Nucl. Phys. B298 (1988) 653

[53] M. L. Mangano, "The Color Structure Of Gluon Emission," Nucl. Phys. B309, 461 (1988)

[54] F. Cachazo and P. Svrcek, "Lectures on Twistor Strings and Perturbative Yang-Mills Theory," hep-th/0504194

[55] M. Mangano and S. J. Parke, "Multiparton Amplitudes In Gauge Theories," Phys. Rep. 200 (1991) 301.

[56] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, "Fusing Gauge Theory Tree Amplitudes Into Loop Amplitudes," Nucl.Phys. B435 (1995) 59-101 [hep-ph/9409265].

[57] Z. Bern and A. G. Morgan, "Massive Loop Amplitudes from Unitarity," Nucl. Phys. B 467, 479 (1996) [hep-ph/9511336].

[58] Z. Bern, L. J. Dixon and D. A. Kosower, "Unitarity-based techniques for one-loop calculations in QCD," Nucl. Phys. Proc. Suppl. 51C, 243 (1996) [hep-ph/9606378].

[59] Z. Bern, L. J. Dixon and D. A. Kosower, "Progress in one-loop QCD computations," Ann. Rev. Nucl. Part. Sci. 46, 109 (1996) [hep-ph/9602280].

[60] S. Catani, "The Singular Behaviour of QCD Amplitudes at Two-loop Order," Phys.Lett. B427 (1998) 161-171 [hep-ph/9802439].

[61] S. Parke and T. Taylor, "An Amplitude For N Gluon Scattering," Phys. Rev. Lett. 56 (1986) 2459

[62] F. A. Berends and W. T. Giele, "Recursive Calculations For Processes With N Gluons," Nucl. Phys. B306 (1988) 759.